The Numerical Solution of Linear Third Order Boundary Value Problems using Nonpolynomial Spline Technique

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Abstract: Second and fourth order convergent methods based on Quartic nonpolynomial spline function are presented for the numerical solution of a third order two-point boundary value problem. The proposed approach gives better approximations than existing polynomial spline and finite difference methods and has a lower computational cost. Convergence analysis of the proposed method is discussed; two numerical examples are included to illustrate the efficiency of the method. [Journal of American Science. 2010;6(12):303-309]. (ISSN: 1545-1003).

Keywords: Quartic nonpolynomial spline; third order two-point boundary value problem; convergence analysis, finite difference.

1. Introduction:

Many problems in mathematical and engineering sciences are formulated in boundary value problems for third order differential equations as in physical oceanography and in the frame work of variational inequality theory and in many branches of pure and applied mathematics. For more details show [1, 2]

We shall consider a numerical solution of the following linear third order two-point boundary value problem

$$y^{(3)}+f(x)\,y=g(x),\; x\in[a,b] \tag{1.1}$$

Subject to the boundary conditions

$$y(a)=k_1,\; y^{(1)}(a)=k_2,\; y^{(1)}(b)=k_3 \tag{1.2}$$

Where $k_i, i=1,2,3$ are finite real constants, the functions $f(x)$ and $g(x)$ are continuous on the interval $[a, b]$, the analytical solution of (1.1) and (1.2) cannot be obtained for arbitrary choices of $f(x)$ and $g(x)$. The numerical analysis literature contains other methods developed to find approximate solutions of these types of boundary value problems. Al-Said and Noor [3, 4] developed a second order method for solving a system of third order two-point boundary value problems using cubic and quartic polynomial spline functions respectively; Al-Said and Noor [5] have developed a second order finite difference method at midpoints. A.Khan and T.Aziz [6] established and discussed convergent fourth order method for this problem with the change in the boundary conditions $y(a)=k_1,\; y^{(1)}(a)=k_2,\; y(b)=k_3$ using quintic polynomial spline functions.

S.ul.Islam et al. [8] have developed a smooth approximation for solving a system of third order obstacle problem based on nonpolynomial spline which provides bases for our method.

In the present paper, Quartic nonpolynomial spline functions are applied to develop a new numerical method for obtaining smooth approximations to the solution of such third-order differential equation. The method is of order two for arbitrary α and β along with $\alpha+\beta=\frac{1}{2}$ and $\alpha\neq 0$, Which will be defined later at the end of the next section and better results will be obtained for choosing α less than β as we will see from the analysis of the local truncation error. And the method of order four for $\alpha=0$ along with $\alpha+\beta=\frac{1}{2}$, in section 2, we derive the consistency relations and develop the quartic nonpolynomial spline method for solving (1.1) subject to (1.2). In section3 and 4 are devoted for the spline solution and convergence analysis of the method. The numerical experiments are given in section 5.

2. Derivation of the method:

We introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into (n+1) equal subintervals where

$$x_i = a + i\,h,\; i = 0, 1, 2, \ldots, n, n+1$$

$$x_0 = a,\; x_{n+1} = b \text{ and } h = \frac{b-a}{n+1} \tag{2.1}$$

Let $y(x)$ be the exact solution of the system (1.1) and (1.2) and S_i be an approximation to $y_i=y(x_i)$ obtained by the spline function $Q_i(x)$ passing through the points (x_i, s_i) and (x_{i+1}, s_{i+1}).

Each quartic nonpolynomial spline segment $Q_i(x)$ has the form:

$$Q_i(x) = a_i \cos k(x - x_i) + b_i \sin k(x - x_i)$$
$$+ c_i(x - x_i)^2 + d_i(x - x_i) + e_i$$
$$, i = 0,1, \dots \dots n \qquad (2.2)$$

Where a_i, b_i, c_i, d_i and e_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method and equation (2.2) reduces to quartic polynomial spline function in $[a,b]$ when $k \to 0$, Choosing the spline function in this form will enable us to generalize other existing polynomial spline methods for arbitrary choices of the parameters α and β which will be defined at the end of this section. Thus, this quartic nonpolynomial spline is now defined by the relations:

$(i)\ S(x) = Q_i(x), x \in [x_i, x_{i+1}], i = 0,1, \dots \dots, n$
$(ii)\ S(x) \in C^\infty [a,b]$ (2.3)

Following the technique of S.ul.Islam et al. [8] we let:

$$Q_i(x) = S_i \quad , Q_i(x_{i+1}) = S_{i+1}$$

$$Q_i^{(1)}(x_i) = D_i \qquad (2.4)$$

$$Q_i^{(3)}(x_i) = T_i , \ Q_i^{(3)}(x_{i+1}) = T_{i+1}$$

For $i = 0,1, \dots\dots\dots,n$, to obtain via a straight forward calculations

$$a_i = h^3 \left[\frac{T_{i+1} - T_i \cos\theta}{\theta^3 \sin\theta}\right]$$
$$b_i = -h^3 \left[\frac{T_i}{\theta^3}\right]$$
$$c_i = \left[\frac{S_{i+1} - S_i}{h^2}\right] - \frac{D_i}{h} - \frac{h}{\theta^2} T_i + \frac{h\,[1 - \cos\theta]\,[T_{i+1} + T_i]}{\theta^3 \sin\theta}$$
$$d_i = D_i + \frac{h^2\, T_i}{\theta^2}$$
$$e_i = S_i - h^3 \left[\frac{T_{i+1} - T_i \cos\theta}{\theta^3 \sin\theta}\right] \qquad (2.5)$$

Where $\theta = kh$, $i = 0, 1, \dots\dots, n$

Using the continuity conditions (ii) and (2.3) of the first and second derivatives at the point (x_i, s_i) that is $Q_{i-1}^{(m)}(x_i) = Q_i^{(m)}(x_i), m = 1, 2$ (2.6)

Using Eqs. (2.2), (2.4), (2.5) and (2.6) yield the relations:

$$D_i + D_{i-1} = \frac{2}{h}[S_i - S_{i-1}] + \frac{2h^2[1 - \cos\theta][T_i + T_{i-1}]}{\theta^3 \sin\theta} - \frac{h^2[T_{i-1} + T_i]}{\theta^2} \qquad (2.7)$$

$$D_i - D_{i-1} = \frac{1}{h}[S_{i-1} - 2S_i + S_{i+1}] + \frac{h^2[1 - \cos\theta]}{\theta^3 \sin\theta}[T_{i+1} - T_{i-1}] + \frac{h^2}{\theta^2}[T_{i-1} - T_i] + \frac{h^2 \cos\theta}{\theta \sin\theta} T_i - \frac{h^2}{2\theta \sin\theta}[T_{i+1} + T_{i-1}] \qquad (2.8)$$

Adding Eqs. (2.7) and (2.8) we get

$$D_i = \frac{1}{2h}[S_{i+1} - S_{i-1}] - \frac{h^2}{\theta^2} T_i + \frac{h^2[1 - \cos\theta]}{2\,\theta^3 \sin\theta}[T_{i+1} + 2T_i + T_{i-1}] + \frac{h^2 \cos\theta}{2\theta \sin\theta} T_i - \frac{h^2}{4\theta \sin\theta}[T_{i+1} + T_{i-1}] \qquad (2.9)$$

Similarly

$$D_{i-1} = \frac{1}{2h}[S_i - S_{i-2}] - \frac{h^2}{\theta^2} T_{i-1} + \frac{h^3[1 - \cos\theta]}{2\,\theta^3 \sin\theta}[T_i + 2T_{i-1} + T_{i-2}] + \frac{h^2 \cos\theta}{2\theta \sin\theta} T_{i-1} - \frac{h^2}{4\theta \sin\theta}[T_i + T_{i-2}] \qquad (2.10)$$

D_i And D_{i-1} are eliminated from equation (2.7) with the help of Eqs. (2.9) and (2.10) to get the following scheme:

$$-S_{i-2} + 3S_{i-1} - 3S_i + S_{i+1} = h^3\left[\alpha\left(T_{i-2} + T_{i+1}\right) + \beta\left(T_{i-1} + T_i\right)\right], i = 2,3,\dots,n-1 \qquad (2.11)$$

Where

$T_i = -f_i S_i + g_i$ *with* $f_i = f(x_i)$ *and* $g_i = g(x_i)$

And
$$\alpha = \left[\frac{1}{2\theta \sin\theta} - \frac{1 - \cos\theta}{\theta^3 \sin\theta}\right]$$
$$\beta = \left[\frac{1 - 2\cos\theta}{2\theta \sin\theta} + \frac{1 - \cos\theta}{\theta^3 \sin\theta}\right]$$

The relation (2.11) gives (n-2) linear algebraic equations in the (n) unknowns S_i , $i = 1, 2, \dots\dots,n$, so we need two more equations, one at each end of the range of integration for direct computation of S_i . Here, for our system (1.1) and (1.2) we also derive these two equations by Taylor series and the method of undetermined coefficients, these equations are:

$$-4S_1+S_2=-3S_0-2hS_0^{(1)}+h^3\left(w_0T_0+w_1T_1+w_2T_2+w_3T_3\right) at\ i=1 \tag{2.12}$$

And

$$-3S_{n-2}+8S_{n-1}-5S_n=-2hS_{n+1}^{(1)}+h^3\left(\sigma_0T_n+\sigma_1T_{n-1}+\sigma_2T_{n-2}+\sigma_3T_{n-3}\right), at\ i=n \tag{2.13}$$

Where w_i's and σ_i's will be determined later to get the required order of accuracy.

The local truncation errors t_i, $i=1,2\ ...n$ associated with the scheme (2.11)– (2.13) can be obtained as follows:

First we rewrite the scheme (2.11) – (2.13) in the form

$$-4y_1+y_2=-3y_0-2hy_0^{(1)}+h^3\left(w_0y_0^{(3)}+w_1y_1^{(3)}+w_2y_2^{(3)}+w_3y_3^{(3)}\right)+t_1, i=1 \tag{2.14}$$

$$-y_{i-2}+3y_{i-1}-3y_i+y_{i+1}=h^3\left[\alpha\left(y_{i-2}^{(3)}+y_{i+1}^{(3)}\right)+\beta\left(y_{i-1}^{(3)}+y_i^{(3)}\right)\right]+t_i, i=2,3,4,...,n-1 \tag{2.15}$$

And

$$-3y_{n-2}+8y_{n-1}-5y_n=-2hy_{n+1}^{(1)}+h^3\left[\sigma_0y_n^{(3)}+\sigma_1y_{n-1}^{(3)}+\sigma_2y_{n-2}^{(3)}+\sigma_3y_{n-3}^{(3)}\right]+t_n, i=n \tag{2.16}$$

The terms $y_{i-2}^{(3)}$, $y_{i+1}^{(3)}$, etc in Eq. (2.15) are expanded around the point x_i using Taylor series and the expressions for t_i , i = 2, ... n-1 can be obtained. Also, expressions for t_i ,i = 1,n are obtained by expanding Eqns. (2.14) and (2.16) around the point x_0 and x_n , respectively, using Taylor series and the expressions for t_i , i = 1, n can be obtained as follow:

$$t_i=\begin{cases} h^3y_0^{(3)}\left[\frac{2}{3}-(w_0+w_1+w_2+w_3)\right]+h^4y_0^{(4)}\left[\frac{1}{2}-(w_1+2w_2+3w_3)\right]+h^5y_0^{(5)}\left[\frac{7}{30}-(\frac{w_1+4w_2+9w_3}{2})\right] \\ +h^6y_0^{(6)}\left[\frac{1}{12}-(\frac{w_1+8w_2+27w_3}{6})\right]+h^7y_0^{(7)}\left[\frac{31}{1260}-(\frac{w_1+16w_2+81w_3}{24})\right]+O\left(h^8\right), i=1 \\ h^3y_i^{(3)}[1-(2\alpha-2\beta)]+h^4y_i^{(4)}\left[(\alpha+\beta)-\frac{1}{2}\right]+h^5y_i^{(5)}\left[\frac{1}{4}-(\frac{5\alpha+\beta}{2})\right] \\ +h^6y_i^{(6)}\left[\frac{-1}{12}+(\frac{7\alpha+\beta}{6})\right]+h^7y_i^{(7)}\left[\frac{1}{40}-(\frac{17\alpha+\beta}{24})\right]+O\left(h^8\right), i=2,...n-1 \\ h^3y_n^{(3)}\left[\frac{11}{3}-(\sigma_0+\sigma_1+\sigma_2+\sigma_3)\right]+h^4y_n^{(4)}\left[\frac{-4}{3}+(\sigma_1+2\sigma_2+3\sigma_3)\right]+h^5y_n^{(5)}\left[\frac{49}{60}-(\frac{\sigma_1+4\sigma_2+9\sigma_3}{2})\right] \\ +h^6y_n^{(6)}\left[\frac{-43}{180}+(\frac{\sigma_1+8\sigma_2+27\sigma_3}{6})\right]+h^7y_n^{(7)}\left[\frac{39}{504}-(\frac{\sigma_1+16\sigma_2+81\sigma_3}{24})\right]+O\left(h^8\right), i=n \end{cases} \tag{2.17}$$

The scheme (2.11) – (2.13) gives rise to a family of methods of different orders as follows:

2.1 Second order method

For arbitrary values of α and β along with $\alpha+\beta=\frac{1}{2}$, $\alpha \neq 0$

$$(w_0, w_1, w_2, w_3)=(\frac{2}{15},\frac{7}{12},\frac{-1}{15},\frac{1}{60})$$

And

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3)=(\frac{157}{60},\frac{19}{30},\frac{11}{20},\frac{-2}{15})$$

Then the local truncation errors given by equation (2.17)are

$$t_i=\begin{cases} \frac{-29}{2520}h^7y_0^{(7)}+O\left(h^8\right), i=1 \\ (-2\alpha)h^5y_i^{(5)}+O\left(h^6\right), i=2,3,...,n-1 \\ \frac{677}{5040}h^7y_n^{(7)}+O\left(h^8\right), i=n \end{cases} \tag{2.18}$$

So, better results occurred for choosing α less than β whose sum is $\frac{1}{2}$

2.2 Fourth order method

For $\alpha=0$ and $\beta=\frac{1}{2}$

$(w_0, w_1, w_2, w_3)=(\frac{2}{15},\frac{7}{12},\frac{-1}{15},\frac{1}{60})$ And $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)=(\frac{157}{60},\frac{19}{30},\frac{11}{20},\frac{-2}{15})$ Then the local truncation errors given by equation (2.17) are

$$t_i = \begin{cases} \dfrac{-29}{2520} h^7 y_0^{(7)} + O\left(h^8\right), & i = 1 \\ \dfrac{1}{240} h^7 y_i^{(7)} + O\left(h^8\right), & i = 2, \ldots\ldots, n-1 \\ \dfrac{677}{5040} h^7 y_n^{(7)} + O\left(h^8\right), & i = n \end{cases} \tag{2.19}$$

Remark

(1) When $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$ then the scheme (2.11) reduces to Al-Said and Noor method based on cubic polynomial spline [3].

(2) When $\alpha = \frac{1}{24}$ and $\beta = \frac{11}{24}$ then the scheme (2.11) reduces to Al-Said and Noor [4], Usmani and Sakai [7] methods based on quartic polynomial spline.

3. Spline solutions:

The spline solution of (1.1) with the boundary condition (1.2) is based on the linear equations given by (2.11) – (2.13).

Let $Y = (y_i)$, $S = (S_i)$, $C = (C_i)$, $T = (T_i)$, $E = (e_i) = Y - S$

Be n-dimensional column vectors, then we can write the standard matrix equations for the nonpolynomial spline method in the form:

(i) $NY = C + T$
(ii) $NS = C$ (3.1)
(iii) $NE = T$

We also have $N = N_0 + h^3 BF;\ F = \mathrm{diag}\left(f_i\right)$, (3.2)

And the matrices N_0 and B are defined by

$$N_0 = \begin{pmatrix} -4 & 1 & & & & & \\ 3 & -3 & 1 & & & & \\ -1 & 3 & -3 & 1 & & & \\ & -1 & 3 & -3 & 1 & & \\ & & & & & & \\ & & & -1 & 3 & -3 & 1 \\ & & & & -3 & 8 & -5 \end{pmatrix} \tag{3.3}$$

The matrix B has the form:

$$B = \begin{pmatrix} w_1 & w_2 & w_3 & & & & \\ \beta & \beta & \alpha & & & & \\ \alpha & \beta & \beta & \alpha & & & \\ & & & & & & \\ & & & \alpha & \beta & \beta & \alpha \\ & & & \sigma_3 & \sigma_2 & \sigma_1 & \sigma_0 \end{pmatrix} \tag{3.4}$$

For the vector C,

$$C_i = \begin{cases} -3k_1 - 2hk_2 + h^3\left[w_0(-f_0 k_1 + g_0) + \sum_{J=1}^{J=3} w_J g_J\right], & i = 1 \\ k_1 - h^3 \alpha k_1 f_0 + h^3\left[\alpha(g_0 + g_3) + \beta(g_1 + g_2)\right], & i = 2 \\ h^3\left[\alpha(g_{i-2} + g_{i+1}) + \beta(g_{i-1} + g_i)\right], & i = 3, 4, \ldots\ldots, n-1 \\ -2h k_3 + h^3\left[\sum_{J=0}^{J=3} \sigma_J g_{n-J}\right], & i = n \end{cases} \tag{3.5}$$

4. Convergence analysis

Our main purpose now is to derive a bound on $\|E\|_\infty$. We now turn back to the error equation (*iii*) in (3.1) and rewrite it in the form

$E = N^{-1}T = \left(N_0 + h^3 BF\right)^{-1} T = \left(I + N_0^{-1} h^3 BF\right)^{-1} N_0^{-1} T$

which implies that:

$$\|E\|_\infty = \left\|I + N_0^{-1} h^3 BF\right\|_\infty \left\|N_0^{-1}\right\|_\infty \|T\|_\infty \tag{4.1}$$

In order to derive the bound on $\|E\|_\infty$, the following two lemmas are needed.

Lemma 4.1, ([9]). If G is a square matrix of order n and $\|G\|_\infty < 1$, then $(I + G)^{-1}$ exists and

$$\left\|(I + G)^{-1}\right\|_\infty < \frac{1}{1 - \|G\|_\infty}$$

Lemma 4.2, the matrix $\left(N_0 + h^3 BF\right)$ is nonsingular, if $\|f\| < \frac{243}{w}$ where:

$$w = 11(b-a)^3\left[2 + \frac{3h^2}{(b-a)^2}\right]$$

Proof. Since,

$N = N_0 + h^3 BF = \left(I + N_0^{-1} h^3 BF\right) N_0$ and the matrix N_0 is nonsingular, so to prove N is nonsingular it is sufficient to show $\left(I + N_0^{-1}h^3BF\right)$ nonsingular. Moreover,

$$\|F\|_\infty \le \|f\| = \max_{a\le x\le b}|f(x)| \quad (4.2)$$

$$\|N_0^{-1}\|_\infty \le \frac{2h^{-3}}{81}\left[(b-a)^3 + \frac{3h^2}{2}(b-a)\right],\ see\ [7] \quad (4.3)$$

$$\|B\|_\infty = \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 = \frac{11}{3} \quad (4.4)$$

Also, $\|N_0^{-1}h^3BF\|_\infty = h^3\|N_0^{-1}\|_\infty \|B\|_\infty \|F\|_\infty$ (4.5)

Therefore, substituting $\|F\|_\infty, \|N_0^{-1}\|_\infty$ *and* $\|B\|_\infty$ in (4.5) we get

$$\|N_0^{-1}h^3BF\|_\infty \le \frac{22}{243}\left[(b-a)^3 + \frac{3h^2(b-a)}{2}\right]\|f\| \quad (4.6)$$

Since, $\|f\| < \dfrac{243}{w}$ (4.7)

Therefore, Eq. (4.7) leads to

$$\|N_0^{-1}h^3BF\|_\infty \le 1 \quad (4.8)$$

From Lemma 4.1, it shows that the matrix N is nonsingular. Since $\|N_0^{-1}h^3BF\|_\infty < 1$, so using Lemma (4.1) and Eq. (4.1) follow that

$$\|E\|_\infty \le \frac{\|N_0^{-1}\|_\infty \|T\|_\infty}{1 - h^3\|N_0^{-1}\|\,\|B\|\,\|F\|} \quad (4.9)$$

From Eq. (2.18) we have

$$\|T\|_\infty = 2\alpha h^5 M_5; M_5 = \max_{a\le x\le b}|y^{(5)}(x)|$$

Then

$$\|E\|_\infty \le \frac{\|N_0^{-1}\|_\infty \|T\|_\infty}{1 - h^3\|N_0^{-1}\|_\infty \|B\|_\infty \|F\|_\infty} \cong O(h^2) \quad (4.10)$$

Also, from Eq. (2.19) we have

$$\|T\|_\infty = \frac{677}{5040}h^7M_7; M_7 = \max_{a\le x\le b}|y^{(7)}(x)|$$

Then

$$\|E\|_\infty \le \frac{\|N_0^{-1}\|_\infty \cdot \|T\|_\infty}{1 - h^3\|N_0^{-1}\|_\infty \|B\|_\infty \|F\|_\infty} \cong O(h^4) \quad (4.11)$$

We summarize the above results in the next theorem.

Theorem 4.1

Let $y(x)$ be the exact solution of the continuous boundary value problem (1.1) with the boundary condition (1.2) and let $y_i, i = 1,2,....n$, satisfy the discrete boundary value problem (*ii*) in (3.1), further, if $e_i = y_i - S_i$ then

1- $\|E\|_\infty \cong O(h^2)$, for second order convergent method

2- $\|E\|_\infty \cong O(h^4)$, for fourth order convergent method

Which are given by (4.10) and (4.11), neglecting all errors due to round off.

5. Numerical examples and discussion:

In this section we illustrate the numerical Techniques discussed in the previous sections by the following two boundary value problems of (1.1) and (1.2), in order to illustrate the comparative Performance of our method (*ii*) in (3.1) over other existing methods. All calculations are implemented by MATLAB 7 .

Example 1:

Consider the boundary value problem

$$y^{(3)} - xy = (x^3 - 2x^2 - 5x - 3)e^x \quad (5.1)$$

$$y(0) = 0,\ y^{(1)}(0) = 1,\ y^{(1)}(1) = -e$$

The analytical solution of (5.1) is

$$y(x) = x(1-x)e^x$$

Example 2

Consider the boundary value problem

$$y^{(3)} + y = (x-4)\sin x + (1-x)\cos x \quad (5.2)$$

$$y(0) = 0,\ y^{(1)}(0) = -1,\ y^{(1)}(1) = \sin(1)$$

The analytical solution of (5.2) is

$$y(x) = (x-1)\sin x$$

The numerical results for our fourth and second orders are summarized in tables 1-4 and compared with the other existing polynomial splines and finite difference methods.

Table 1: ***The observed maximum absolute errors for Example 1***

h	*Fourth order method* $\alpha = 0$, $\beta = \frac{1}{2}$	*Second order method* $\alpha = \frac{1}{128}$, $\beta = \frac{1}{2} - \alpha$
$\frac{1}{16}$	5.2992 – 7	1.5540 – 4 [a]
$\frac{1}{32}$	2.6127 – 8	4.1551 – 5
$\frac{1}{64}$	1.4999 – 9	1.0575 – 5
$\frac{1}{128}$	8.9762 – 11	2.6562 – 6

[a] 1.5540 – 4 = $1.5540*10^{-4}$

Table 2: ***The observed maximum absolute errors for Example 1***

h	**Our fourth order method** $\alpha = 0, \beta = \frac{1}{2}$	**Islam et al. [8]**	**Al-Said and Noor [5]**	**Al-Said and Noor [4]**	**Al-Said and Noor [3]**
$\frac{1}{16}$	5.2992-7	2.1974-5	8.1224-4	8.3597-4	1.6861-3
$\frac{1}{32}$	2.6127-8	1.6192-6	2.1812-4	2.2207-4	4.4510-4
$\frac{1}{64}$	1.4999-9	1.1006-7	5.5859-5	5.6432-5	1.1293-4
$\frac{1}{128}$	8.9762-11	7.1764-9	1.4091-5	1.4168-5	2.8340-5

Table 3: ***The observed maximum absolute errors for Example 2***

h	*Fourth order method* $\alpha = 0$, $\beta = \frac{1}{2}$	*Second order method* $\alpha = \frac{1}{128}$, $\beta = \frac{1}{2} - \alpha$
$\frac{1}{16}$	2.3819 – 8	9.0774 – 6
$\frac{1}{32}$	1.1184 – 9	2.4289 – 6
$\frac{1}{64}$	6.3020 – 11	6.1842 – 7
$\frac{1}{128}$	3.7640 – 12	1.5534 – 7

Table 4: ***The observed maximum absolute errors for Example 2***

h	Our fourth order method $\alpha = 0, \beta = \frac{1}{2}$	Islam et al. [8]	Al-Said and Noor [5]	Al-Said and Noor [4]	Al-Said and Noor [3]
$\frac{1}{16}$	2.3819–8	9.2517–7	4.5978–5	4.8237–5	9.7501–5
$\frac{1}{32}$	1.1184–9	6.8079–8	1.2530–5	1.2948–5	2.5965–5
$\frac{1}{64}$	6.3020–11	4.5822–9	3.2356–6	3.2980–6	6.6004–6
$\frac{1}{128}$	3.7640–12	2.9515–10	8.1999–7	8.284–7	1.6573–6

It is verified from the Tables 1- 4 that on reducing the step size from h to $\frac{h}{2}$ the maximum error $\|E\|$ is approximately reduced by a factor $\frac{1}{2^p}$, where p is the order of the method which confirms that our method is a second and fourth orders convergent as predicted in section 4.

6. Conclusion:

Two new methods are presented for solving third order two-point boundary value problem using quartic nonpolynomial spline functions. These methods are shown to be optimal second and optimal fourth orders which have better accuracy compared with Al-Said and Noor [3-5] and S.ul.Islam et al [8]. The obtained numerical results show that the proposed methods maintain a very remarkable high accuracy which make them are very encouraging for dealing with the solution of two-point boundary value problems.

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